

A LIE-THEORETIC CONSTRUCTION OF SPHERICAL SYMPLECTIC REFLECTION ALGEBRAS

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ABSTRACT. We propose a construction of the spherical subalgebra of a symplectic reflection algebra of an arbitrary rank corresponding to a star-shaped affine Dynkin diagram. Namely, it is obtained from the universal enveloping algebra of a certain semi-simple Lie algebra by the process of quantum Hamiltonian reduction. As an application, we propose a construction of finite-dimensional representations of the spherical subalgebra.

INTRODUCTION

The main result of this paper is the realization of the spherical subalgebra of the wreath product symplectic reflection algebra of rank n of types D_4, E_6, E_7, E_8 , introduced in [EG], as a quantum Hamiltonian reduction of the tensor product of m quotients of the enveloping algebra $U(\mathfrak{sl}_{n\ell})$, where ℓ is 2, 3, 4, and 6, and m is 4, 3, 3, and 3, respectively. This allows one to define a functor which attaches a representation of the spherical symplectic reflection algebra to a collection of m representations of $\mathfrak{sl}_{n\ell}$, which are annihilated by certain ideals. In particular, this gives an explicit Lie-theoretic construction of many finite dimensional representations of spherical symplectic reflection algebras, most of which appear to be new. In the rank 1 case, all finite dimensional representations of spherical symplectic reflection algebras are classified by Crawley-Boevey and Holland ([CBH]) and our construction yields several explicit Lie-theoretic realizations of all of them.

The proof of the main result is based on the previous work [EGGO], in which the spherical subalgebra of a wreath product symplectic reflection algebra (of any type) is realized as the quantum Hamiltonian reduction from the algebra of differential operators on representations (with a certain dimension vector) of the Calogero-Moser quiver, obtained from the corresponding extended Dynkin quiver by adjoining an auxiliary vertex, linked to the extending vertex. Namely, in the case when the extended Dynkin graph is star-shaped (which happens in the cases D_4, E_6, E_7, E_8), this reduction can be performed in 2 steps: first the reduction with respect to the groups of basis changes at the non-central vertices, and then with respect to the group of basis changes at the central (branching) vertex of the star. By the localization theorem for partial flag varieties, after the first step, we obtain a

tensor product of m quotients of enveloping algebras (the factors correspond to the branches of the extended Dynkin diagram), which yields the result.

Recall that by the results of [EG], a wreath product spherical symplectic reflection algebra can be viewed as a quantization of the wreath product Calogero-Moser space, which is a deformation of the n -th Hilbert scheme of the resolution of a Kleinian singularity. Our main result is a quantum analog of the statement from classical symplectic geometry, stating that this wreath product Calogero-Moser space may be obtained by classical Hamiltonian reduction from a product of m coadjoint orbits of the Lie algebra $\mathfrak{sl}_{n\ell}$, or, equivalently, as the space of solutions of a special kind of the additive Deligne-Simpson problem. This classical result can be found in [EGO], Section 2.6.

We expect that the main result of this paper has a q -deformed analog, in which the Lie algebra $\mathfrak{sl}_{n\ell}$ is replaced by the corresponding quantum group $U_q(\mathfrak{sl}_{n\ell})$, and spherical symplectic reflection algebras are replaced by spherical subalgebras of generalized double affine Hecke algebras (DAHAs), introduced for higher rank in [EGO] and for rank 1 in [EOR] (except the D_4 case, where they were known before due to the work of Sahi and Stokman, [Sa, St]). Such a result would be a quantization of the result of Section 5.2 of [EGO], which is a multiplicative version of the abovementioned result from Section 2.6, as it is concerned with the multiplicative, rather than additive, Deligne-Simpson problem. In fact, we can construct the corresponding functor, which attaches a representation of the spherical subalgebra of the generalized DAHA to a collection of m representations of the quantum group annihilated by certain ideals. This, as well as non-spherical versions of the results of this paper, will be discussed in future publications.

The paper is organized as follows. In Section 1, we introduce the notation and formulate the main result (Theorem 1.4.1). We prove it in Section 2. In Section 3, we use our construction to produce finite dimensional representations of spherical symplectic reflection algebras, starting from known classes of Lie algebra representations.

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1. THE MAIN THEOREM

1.1. Quantum Hamiltonian Reduction. The following construction is the quantum analog of the Hamiltonian reduction procedure.

Let A be an associative algebra, \mathfrak{g} be a Lie algebra, $\mu : \mathfrak{g} \rightarrow A$ be a homomorphism of Lie algebras.

Definition 1.1.1. Define the associative algebra

$$\mathfrak{A}(A, \mathfrak{g}, \mu) = (A/A\mu(\mathfrak{g}))^{\mathfrak{g}},$$

where the invariants are taken with respect to the adjoint action of \mathfrak{g} on A . This algebra is called the **quantum Hamiltonian reduction** of A with respect to \mathfrak{g} with quantum moment map μ .

The following proposition is well known, but we give its proof for reader's convenience. It summarizes the main properties of the quantum Hamiltonian reduction, and gives a construction of its representations.

Proposition 1.1.2. Assume that \mathfrak{g} is reductive, and the adjoint action of \mathfrak{g} on A is completely reducible. Then:

(i)

$$\mathfrak{A}(A, \mathfrak{g}, \mu) = A^{\mathfrak{g}}/(A\mu(\mathfrak{g}))^{\mathfrak{g}} = A^{\mathfrak{g}}/(\mu(\mathfrak{g})A)^{\mathfrak{g}} = (A/\mu(\mathfrak{g})A)^{\mathfrak{g}}.$$

(ii) If V is any A -module then $\mathfrak{A}(A, \mathfrak{g}, \mu)$ acts naturally on the cohomology $H^i(\mathfrak{g}, V)$ and the homology $H_i(\mathfrak{g}, V)$, in particular, on the invariants $V^{\mathfrak{g}}$ and the coinvariants $V_{\mathfrak{g}}$.

(iii) Suppose that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, so $\mu = \mu_1 \oplus \mu_2$. Then μ_2 descends to a map $\overline{\mu}_2 : \mathfrak{g}_2 \rightarrow \mathfrak{A}(A, \mathfrak{g}_1, \mu_1)$ and we have

$$\mathfrak{A}(A, \mathfrak{g}, \mu) \cong \mathfrak{A}(\mathfrak{A}(A, \mathfrak{g}_1, \mu_1), \mathfrak{g}_2, \overline{\mu}_2).$$

Proof. Let us prove (i). Since the functor of taking \mathfrak{g} -invariants is exact on completely reducible modules, the only equality we need to prove is (using a shorthand notation) $(A\mu(\mathfrak{g}))^{\mathfrak{g}} = (\mu(\mathfrak{g})A)^{\mathfrak{g}}$.

Since \mathfrak{g} is reductive and A is completely reducible as a \mathfrak{g} -module, the space $(A\mu(\mathfrak{g}))^{\mathfrak{g}}$ is the image of $(A \otimes \mathfrak{g})^{\mathfrak{g}}$ under the multiplication map. Therefore, any element $x \in (A\mu(\mathfrak{g}))^{\mathfrak{g}}$ is the image of some element $x' = \sum a_i \otimes b_i \in (A \otimes \mathfrak{g})^{\mathfrak{g}}$, where b_i is a basis of \mathfrak{g} . Let $[b_i b_j] = \sum c_{ij}^k b_k$. Then

$$\sum [b_j a_i] \otimes b_i + \sum a_i \otimes [b_j b_i] = 0,$$

which implies that $[b_j a_i] = \sum c_{pj}^i a_p$. Thus $\sum_i [b_i a_i] = \sum_{p,i} c_{pi}^i a_p = 0$, as $\sum c_{pi}^i = \text{Tr}(\text{ad}(b_p)) = 0$ (because \mathfrak{g} is reductive). So $x = \sum a_i b_i = \sum b_i a_i \in (\mu(\mathfrak{g})A)^{\mathfrak{g}}$, as desired.

To prove (ii), note that $A/A\mu(\mathfrak{g})$ obviously acts by operators from $V^{\mathfrak{g}}$ to V , so $(A/A\mu(\mathfrak{g}))^{\mathfrak{g}}$ acts on $V^{\mathfrak{g}}$. Also, it is clear that $A/\mu(\mathfrak{g})A$ acts from V to $V_{\mathfrak{g}}$, so $(A/\mu(\mathfrak{g})A)^{\mathfrak{g}}$ acts on $V_{\mathfrak{g}}$. These actions are functorial in V , so they extend to derived functors, as desired.

Statement (iii) follows easily from the fact that \mathfrak{g} is reductive and acts completely reducibly on A . \square

1.2. Symplectic Reflection Algebras. Let L be a 2-dimensional complex vector space equipped with a symplectic form ω . Let Γ be a finite subgroup of $Sp(L) \cong SL_2(\mathbb{C})$. Let $\Gamma_n = S_n \ltimes \Gamma^n$, where S_n is the symmetric group.

Let $\mathbb{C}[\Gamma]$ be the group algebra of Γ , and $Z\Gamma$ be the center of $\mathbb{C}[\Gamma]$. Let $c : \Gamma \setminus \{1\} \rightarrow \mathbb{C}$ be a conjugation invariant function.

For $u \in L$ denote by $u_l \in L^n$ the corresponding element in the l -th summand. Similarly, for $\gamma \in \Gamma$ let γ_l be the element of $\Gamma^n \subset \Gamma_n$, in which γ stands in the l -th place. Let $s_{lm} \in S_n$ be the transposition of l and m . Let $k, t \in \mathbb{C}$.

Definition 1.2.1. *The symplectic reflection algebra $H_{t,k,c} = H_{t,k,c}(\Gamma_n)$ of rank n associated to Γ is the quotient of the smash product $\mathbb{C}[\Gamma_n] \ltimes T(L^n)$ (where the group algebra $\mathbb{C}[\Gamma_n]$ acts on $T(L^n)$ in the obvious way), by the additional relations*

$$[u_l, v_m] = -\frac{k}{2} \sum_{\gamma \in \Gamma} \omega(\gamma u, v) s_{lm} \gamma_l \gamma_m^{-1}, \quad u, v \in L, \quad l \neq m;$$

$$[u_l, v_l] = \omega(u, v) \left(t + \sum_{\gamma \in \Gamma, \gamma \neq 1} c_\gamma \gamma_l + \frac{k}{2} \sum_{m \neq l} \sum_{\gamma \in \Gamma} s_{lm} \gamma_l \gamma_m^{-1} \right).$$

Note that for any $a \neq 0$ we have $H_{at,ak,ac} \cong H_{t,k,c}$, so there are two essentially different cases: the *classical* case $t = 0$ and the *quantum* case $t = 1$. In this paper, we focus on the quantum case $t = 1$.

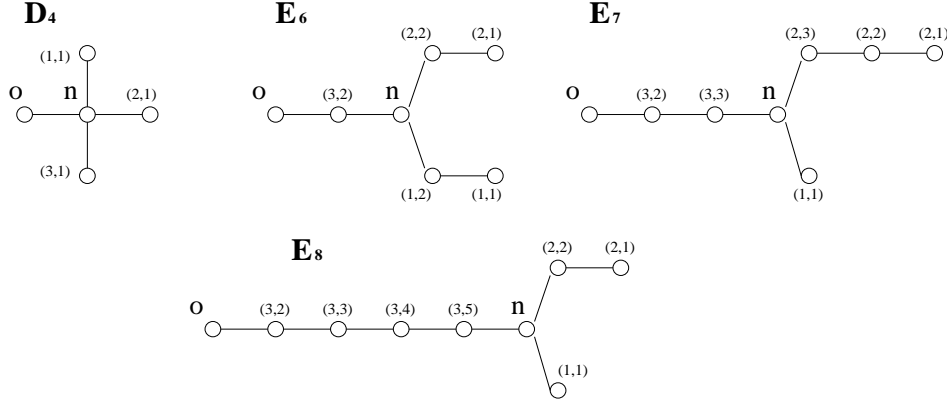
1.3. Enveloping algebra quotients. Let \mathfrak{g} be a reductive Lie algebra. Let \mathfrak{p} be a parabolic subalgebra in \mathfrak{g} , and let $\mu : \mathfrak{p} \rightarrow \mathbb{C}$ be a Lie algebra character. Let $\text{Ker} \mu \subset U(\mathfrak{p}) \subset U(\mathfrak{g})$ be the kernel of the 1-dimensional representation $\mu : U(\mathfrak{p}) \rightarrow \mathbb{C}$, and $I(\text{Ker} \mu)$ be the largest two-sided ideal in $U(\mathfrak{g})$ contained in the left ideal $U(\mathfrak{g}) \cdot \text{Ker} \mu$. Define the algebra

$$U_\mu^{\mathfrak{p}}(\mathfrak{g}) = U(\mathfrak{g})/I(\text{Ker} \mu).$$

Let $M_\mu^{\mathfrak{p}} = \text{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mu$ be the generalized Verma module. It is easy to show that the ideal $I(\text{Ker} \mu)$ coincides with the annihilator $\text{Ann}(M_\mu^{\mathfrak{p}})$. Therefore, the algebra $U_\mu^{\mathfrak{p}}(\mathfrak{g})$ acts on any representation of \mathfrak{g} with highest weight μ with respect to \mathfrak{p} .

1.4. The Main Theorem. Let D be a graph with a shape of a star, that is, a tree with one m -valent vertex \mathbf{n} , called the *node* or the *branching vertex*, and the rest of the vertices 2- and 1-valent (which form m “legs” growing from the node). We label the legs of D by numbers $1, \dots, m$, and let d_i be the number of vertices in the i -th leg including the node. Suppose that $d_1 \leq \dots \leq d_m$. Then let us enumerate the vertices (excluding \mathbf{n}) by pairs (j, i) , where $1 \leq j \leq m$ is the leg number, and $1 \leq i < d_j$ is the number of our vertex on the leg starting from the outside (so $(j, d_j - 1)$ is connected with \mathbf{n}).

Recall that via the McKay correspondence, Γ corresponds to a finite ADE Dynkin diagram D_Γ^0 , and the irreducible finite-dimensional representations of Γ correspond to vertices of the affinization D_Γ of D_Γ^0 . Suppose that D_Γ has the shape of a star. It means that D_Γ^0 is the Dynkin graph of D_4 or E_6 or E_7 or E_8 . Then the sets (d_1, \dots, d_m) in the four cases under consideration are: $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, and $(2,3,6)$. Let $\ell = d_m$, and note that in our setting all d_i are divisors of ℓ . Denote by $\mathbf{o} = (m, 1)$ the affinizing vertex, that is, the vertex belonging to D_Γ but not D_Γ^0 .



For a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ and a character μ of \mathfrak{p} set

$$\iota_\mu^{\mathfrak{p}} : \mathfrak{g} \hookrightarrow U(\mathfrak{g}) \rightarrow U_\mu^{\mathfrak{p}}(\mathfrak{g}).$$

Denote by $\iota_{\mu_1, \dots, \mu_m}^{\mathfrak{p}_1, \dots, \mathfrak{p}_m}$ the natural map

$$\sum_{i=1}^m 1^{\otimes i-1} \otimes \iota_{\mu_i}^{\mathfrak{p}_i} \otimes 1^{\otimes m-i} : \mathfrak{g} \rightarrow U_{\mu_1}^{\mathfrak{p}_1}(\mathfrak{g}) \otimes \dots \otimes U_{\mu_m}^{\mathfrak{p}_m}(\mathfrak{g}).$$

Let $r > 1$ be a positive integer. For any positive integer s dividing r , denote by $\mathfrak{p}(s, r) \subset \mathfrak{sl}_r$ the parabolic subalgebra generated by the subalgebra \mathfrak{b} of upper triangular matrices, and the elements $f_i = E_{i+1, i}$ for i not divisible by r/s . Also, denote by $\mathfrak{p}'(s, r) \subset \mathfrak{p}(s, r) \subset \mathfrak{sl}_r$ the parabolic subalgebra generated by \mathfrak{b} and f_i for $i > 1$ not divisible by r/s .

Denote by P_Γ the vector space spanned by the roots of the affine root system corresponding to Γ . For each vertex \mathbf{i} of D_Γ denote by $\alpha_{\mathbf{i}} \in P_\Gamma$ the corresponding simple root. For a vertex $\mathbf{i} \in D_\Gamma$, denote by $N_{\mathbf{i}}$ the corresponding representation of Γ .

For a weight $\lambda \in P_\Gamma$ set

$$\mu_j(n, \lambda) = \sum_{i=1}^{d_j-1} \left(\lambda_{(j, i)} - \frac{n\ell}{d_j} \right) \omega_{n \frac{\ell}{d_j} i},$$

where ω_i is the i -th fundamental weight of $\mathfrak{sl}_{n\ell}$.

Let $c : \Gamma \setminus \{1\}$ be a conjugation invariant function. Define the weight $\lambda(c) \in P_\Gamma$ by

$$\lambda(c) = \sum_{\mathbf{i}} \lambda(c)_{\mathbf{i}} \alpha_{\mathbf{i}},$$

where

$$\lambda(c)_{\mathbf{i}} = \frac{1}{|\Gamma|} \left(\dim N_{\mathbf{i}} + \sum_{\gamma \neq 1} c_\gamma \operatorname{Tr}_{N_{\mathbf{i}}}(\gamma) \right).$$

Theorem 1.4.1. *Let*

$$\mathbf{p}_i = \mathbf{p}(d_i, n\ell), \quad \mu_i = \mu_i(n, \lambda(c)) \quad \text{for } 1 \leq i < m,$$

$$\mathbf{p}_m = \mathbf{p}'(d_m, n\ell), \quad \mu_m = \mu_m(n, \lambda(c)) + n \left(\frac{k}{2} - 1 \right) \omega_1 - \frac{k}{2} \omega_n.$$

Then we have

$$eH_{1,k,ce} \cong \mathfrak{A} \left(U_{\mu_1}^{\mathbf{p}_1}(\mathfrak{sl}_{n\ell}) \otimes \cdots \otimes U_{\mu_m}^{\mathbf{p}_m}(\mathfrak{sl}_{n\ell}), \mathfrak{sl}_{n\ell}, \iota_{\mu_1, \dots, \mu_m}^{\mathbf{p}_1, \dots, \mathbf{p}_m} \right).$$

This theorem is a quantum analog of the result of Section 2.6 in [EGO]. We prove it in the next section.

Corollary 1.4.2. *Let Y be a representation of $\otimes_{i=1}^m U_{\mu_i}^{\mathbf{p}_i}(\mathfrak{sl}_{n\ell})$ (for example, $Y = U_1 \otimes \cdots \otimes U_m$, where U_i are representations of $U_{\mu_i}^{\mathbf{p}_i}(\mathfrak{sl}_{n\ell})$, $i = 1, \dots, m$). Then $eH_{1,k,ce}$ acts on $Y^{\mathfrak{sl}_{n\ell}}$ and $Y_{\mathfrak{sl}_{n\ell}}$. More generally, it acts on the cohomology $H^i(\mathfrak{sl}_{n\ell}, Y)$ and the homology $H_i(\mathfrak{sl}_{n\ell}, Y)$.*

Proof. The corollary follows immediately from Theorem 1.4.1 and Proposition 1.1.2. \square

2. PROOF OF THEOREM 1.4.1

2.1. Twisted differential operators. One of the important applications of the quantum Hamiltonian reduction construction is the construction of the sheaf of differential operators on an algebraic variety, twisted by a collection of line bundles.

Namely, let X be a smooth affine complex algebraic variety, $\mathcal{D}(X)$ be the algebra of differential operators on X , and let L_1, \dots, L_d be line bundles on X . Let E denote the total space of the principal $(\mathbb{C}^\times)^d$ -bundle corresponding to $L_1 \oplus \cdots \oplus L_d$. Denote by \mathcal{E}_i the Euler vector field on E along the i -th factor of the fiber, and by \mathcal{E} the map $\mathbb{C}^d \rightarrow \operatorname{Vect}(E)$ sending the standard basis vectors to \mathcal{E}_i .

Let $\chi \in (\mathbb{C}^d)^*$.

Definition 2.1.1. *The algebra of twisted differential operators $\mathcal{D}_{\chi, L_1, \dots, L_d}(X)$ is defined by*

$$\mathcal{D}_{\chi, L_1, \dots, L_d}(X) := \mathfrak{A}(\mathcal{D}(E), \mathbb{C}^d, \mathcal{E} - \chi),$$

This is a flat d -parametric deformation of the algebra $\mathcal{D}(X) \cong \mathcal{D}_{0, L_1, \dots, L_d}(X)$.

Remark 2.1.2. Let ∇_i be connections on L_i with curvatures F_i . Using these connections, we can naturally lift any vector field v on X to a $(\mathbb{C}^\times)^d$ -invariant vector field $\nabla(v)$ on E , and we have

$$(2.1.3) \quad [\nabla(v), \nabla(w)] = \nabla([v, w]) + \sum_{i=1}^d \chi_i F_i(v, w)$$

modulo the ideal generated by $\mathcal{E}_i - \chi_i$. Thus the algebra $\mathcal{D}_{\chi, L_1, \dots, L_d}(X)$ is generated by regular functions $f \in \mathcal{O}(X)$ and elements $\nabla(v)$, $v \in \text{Vect}(X)$, with relations

$$\nabla(fv) = f\nabla(v), [\nabla(v), f] = L_v f,$$

and (2.1.3), where L_v is the Lie derivative. This is the usual definition of twisted differential operators.

If X is smooth but not affine, then for any affine open set $U \subset X$, we have the algebra

$$\mathcal{D}_{\chi, L_1, \dots, L_d}(U) = \mathfrak{A}(\mathcal{D}(E(U)), \mathbb{C}^d, \mathcal{E} - \chi),$$

where $E(U)$ is the preimage of U under the map $E \rightarrow X$. One can show that this defines a quasicoherent sheaf $\mathcal{D}_{\chi, L_1, \dots, L_d}$ on X . This sheaf is called *the sheaf of twisted differential operators*.

Remark 2.1.4. For more on twisted differential operators, see [BB].

2.2. Twisted differential operators on partial flag varieties. Let G be an algebraic group corresponding to \mathfrak{g} , and P the parabolic subgroup of G corresponding to \mathfrak{p} . Then to any Lie algebra character $\mu : \mathfrak{p} \rightarrow \mathbb{C}$ we can canonically attach the algebra of global twisted differential operators $\mathcal{D}_\mu(G/P)$, where the twisting is with respect to the line bundles L_i corresponding to some generators χ_i of the character group $\text{Hom}(P, \mathbb{C}^\times)$. Namely,

$$\mathcal{D}_\mu(G/P) = \mathcal{D}_{\mu^1, \dots, \mu^s, L_1, \dots, L_s}(G/P),$$

where $\mu = \sum \mu^i d\chi_i$, and $s = \dim P/[P, P]$ (it is easy to check that this construction is independent on the choice of generators).

It follows from the definition of twisted differential operators that the natural homomorphism $U(\mathfrak{g}) \rightarrow \mathcal{D}_\mu(G/P)$ defines a homomorphism $i_\mu : U_\mu^\mathfrak{p}(\mathfrak{g}) \rightarrow \mathcal{D}_\mu(G/P)$.

In general, the map i_μ is not always an isomorphism. However, we have the following well known result. Recall that the algebras $U_\mu^\mathfrak{p}(\mathfrak{g})$ and $\mathcal{D}_\mu(G/P)$ have natural filtrations (the first one is induced from the filtration on $U(\mathfrak{g})$, and the second one is by order of differential operators).

Proposition 2.2.1. *If $G = SL_r$ then*

(i) *$\text{gr}(\mathcal{D}_\mu(G/P)) = \mathbb{C}[T^*(G/P)]$, and i_μ is an isomorphism of filtered algebras for any μ ;*

(ii) *for every irreducible finite dimensional representation Y of G , the multiplicity of Y in $\mathcal{D}_\mu(G/P)$ equals the dimension of Y^L , where L is the Levi subgroup of P .*

This is a parabolic version of the localization theorem. For convenience of the reader, let us recall the proof of this result.

Proof. (i) Let $O_{\mathfrak{p}}$ the closure of the adjoint orbit $\text{Ad}(SL_r)n_{\mathfrak{p}}$, where $n_{\mathfrak{p}}$ is a generic element of $\text{rad } \mathfrak{p}$. According to [KP], $O_{\mathfrak{p}}$ is a normal variety. Moreover, we have a natural Springer map $\xi_0 : T^*(G/P) \rightarrow O_{\mathfrak{p}}$ (the moment map for the group action), which is a resolution of singularities in this case (see [BB], 2.7). Indeed, each point of $T^*(G/P)$ can be regarded as a pair (\mathfrak{p}', n) , where \mathfrak{p}' is a parabolic subalgebra conjugate to \mathfrak{p} , and $n \in \text{rad } \mathfrak{p}'$, and the moment map sends (\mathfrak{p}', n) to $n \in O_{\mathfrak{p}}$. Thus, we just need to show that for generic $n \in O_{\mathfrak{p}}$, there is a unique \mathfrak{p}' conjugate to \mathfrak{p} such that $n \in \text{rad } \mathfrak{p}'$; this follows from elementary linear algebra.

Thus, the natural map $\xi_0^* : \mathbb{C}[O_{\mathfrak{p}}] \rightarrow \mathbb{C}[T^*(G/P)]$ induced by the moment map ξ_0 is an isomorphism of graded algebras. Hence we can apply Theorem 5.6 from [BB] (as one of the equivalent conditions stated in this theorem, namely, condition (iii), holds).

By Theorem 5.6(v) of [BB], the associated graded for $U_{\mu}^{\mathfrak{p}}(\mathfrak{g})$ is the algebra $\mathbb{C}[O_{\mathfrak{p}}]$ of polynomial functions on $O_{\mathfrak{p}}$. At the same time, the associated graded algebra of $\mathcal{D}_{\mu}(G/P)$ is contained in the algebra $\mathbb{C}[T^*(G/P)]$ of polynomial functions on the cotangent bundle. Also, it is clear that $\text{gr}(i_{\mu}) = \xi_0^*$. Thus i_{μ} is an isomorphism of filtered algebras, and $\text{gr}(\mathcal{D}_{\mu}(G/P)) = \mathbb{C}[T^*(G/P)]$, as desired.

(ii) It suffices to show that $\mathbb{C}[O_{\mathfrak{p}}]$ has the stated property. There exists a 1-parameter family of semisimple orbits $O_{\mathfrak{p}}^t, t \in \mathbb{C}^{\times}$, which degenerates at $t = 0$ into $O_{\mathfrak{p}}$. These orbits have the form G/L . Thus, as a G -module, $\mathbb{C}[O_{\mathfrak{p}}] = \mathbb{C}[G/L]$, and the statement follows from the Peter-Weyl theorem. \square

2.3. Relation to Quivers. Let Q be a quiver with the set of vertices I . For any edge $a \in Q$ denote by $h(a)$ and $t(a)$ its head and tail, respectively.

For a dimension vector $\beta \in \mathbb{Z}_+^I$ consider the space of representations

$$\text{Rep}_{\beta}(Q) = \bigoplus_{a \in Q} \text{Hom} \left(\mathbb{C}^{\beta_{t(a)}}, \mathbb{C}^{\beta_{h(a)}} \right).$$

This space admits a natural linear action \mathfrak{J} of the group $GL(\beta) = \prod_{i \in I} GL_{\beta_i}$. Denote by $\mathfrak{gl}(\beta)$ the Lie algebra of this group; then we have the corresponding map $d\mathfrak{J} : \mathfrak{gl}(\beta) \rightarrow \mathcal{D}(\text{Rep}_{\beta}(Q))$.

Now let us introduce quiver-related twisted differential operators. For $\chi \in \mathbb{C}^I$ let

$$\chi \text{Tr} = \sum_{i \in I} \chi_i \cdot \text{Tr}_i : \mathfrak{gl}(\beta) \rightarrow \mathbb{C},$$

where Tr_i is the trace on \mathfrak{gl}_{β_i} .

Definition 2.3.1. *Let*

$$\mathcal{D}_{\chi}(Q, \beta) = \mathfrak{A} \left(\mathcal{D}(\text{Rep}_{\beta}(Q)), \mathfrak{gl}(\beta), d\mathfrak{J} - \chi \text{Tr} \right).$$

Now let Γ be of type D_4, E_6, E_7, E_8 . Let D_Γ^{CM} be the graph obtained from the graph D_Γ by adding a vertex \mathbf{s} with one edge from \mathbf{s} to the affinizing vertex \mathbf{o} . Let Q_{CM} be a quiver obtained by orienting all edges of D_Γ^{CM} in some way, so that the additional edge is oriented from \mathbf{s} to \mathbf{o} (the Calogero-Moser quiver, see [EGGO]).

Let $I_{CM} = I \cup \{\mathbf{s}\}$ be the set of vertices of the Calogero-Moser quiver. Introduce the vector $\partial \in \mathbb{C}^I$ by

$$\partial_{\mathbf{i}} = n \left(-\delta_{\mathbf{i}} + \sum_{a \in Q_{CM}: t(a)=\mathbf{i}} \delta_{h(a)} \right),$$

where $\delta := \sum \dim N_{\mathbf{i}} \alpha_{\mathbf{i}}$ is the basic imaginary root.

Proposition 2.3.2. *For the orientation of all edges towards the node we have $\partial_{(i,j)} = n\ell/d_i$.*

Proof. The proof is by a direct computation. \square

Define $\chi^{CM} \in \mathbb{C}^{I_{CM}}$ by

$$\chi_{\mathbf{s}}^{CM} = n \left(\frac{k}{2} - 1 \right), \quad \chi_{\mathbf{o}}^{CM} = \lambda(c)_{\mathbf{o}} - \partial_{\mathbf{o}} - \frac{k}{2}, \quad \chi_{\mathbf{i}}^{CM} = \lambda(c)_{\mathbf{i}} - \partial_{\mathbf{i}}, \quad \mathbf{i} \neq \mathbf{o}, \mathbf{s}.$$

Theorem 2.3.3. [EGGO] *Take $\alpha^{CM} = \alpha_{\mathbf{s}} + n\delta$. Then for any orientation of the quiver we have an isomorphism of filtered algebras*

$$\mathcal{D}_{\chi^{CM}}(Q_{CM}, \alpha^{CM}) \cong eH_{1,k,c}e.$$

2.4. Relation to partial flag varieties. Let $0 < r_1 < r_2 < \dots < r_s < r$ be a collection of positive integers. Denote by $\mathcal{F}\ell(r_1, \dots, r_s; r)$ the corresponding partial flag variety, i.e., the configuration space of s subspaces

$$V_1 \subset \dots \subset V_s \subset V_{s+1} = \mathbb{C}^r$$

such that $\dim V_j = r_j, j \leq s$.

Denote by L_j the line bundle $\wedge^{r_j}(V_j)$. These bundles can be considered as generators of the Picard group of $\mathcal{F}\ell(r_1, \dots, r_s; r)$. Then for a vector $\chi = (\chi_1, \dots, \chi_s)$ one can define the algebra of twisted differential operators on $\mathcal{D}_{\chi, L_1, \dots, L_s}(\mathcal{F}\ell(r_1, \dots, r_s; r))$ following Definition 2.1.1. Abbreviating the notation, we will denote this algebra by $\mathcal{D}_{\chi}(\mathcal{F}\ell(r_1, \dots, r_s; r))$.

Now let $Y = \bigoplus_{i=1}^s \text{Hom}(V_i, V_{i+1})$, and $H = \prod_{i=1}^s GL(V_i)$. Let \mathfrak{J} be the natural representation of H on Y . Note that the action of the remaining GL_r commutes with \mathfrak{J} . Denote by $d\mathfrak{J}$ the corresponding map from the Lie algebra $\text{Lie} H$ to the algebra of differential operators on Y . For $\chi = (\chi_1, \dots, \chi_s) \in \mathbb{C}^s$ let $\chi \text{Tr} = \sum \chi_i \cdot \text{Tr}_i$ be a character of this Lie algebra.

Theorem 2.4.1. *Let $\chi = (\chi_1, \dots, \chi_s) \in \mathbb{C}^s$. Then we have a GL_r -equivariant isomorphism of filtered algebras*

$$\mathfrak{A} \left(\mathcal{D} \left(\bigoplus_{i=1}^s \text{Hom}(V_i, V_{i+1}) \right), \bigoplus_{i=1}^s \mathfrak{gl}_{r_i}, d\mathfrak{J} - \chi \text{Tr} \right) \cong \mathcal{D}_{\chi}(\mathcal{F}\ell(r_1, \dots, r_s; r)).$$

Proof. Denote by $\text{Inc}(V_i, V_{i+1}) \subset \text{Hom}(V_i, V_{i+1})$ the open set consisting of inclusions. We have a natural homomorphism

$$\mathcal{D} \left(\bigoplus_{i=1}^s \text{Hom}(V_i, V_{i+1}) \right) \rightarrow \mathcal{D} \left(\prod_{i=1}^s \text{Inc}(V_i, V_{i+1}) \right),$$

which is in fact an isomorphism, because the set of non-inclusions has codimension ≥ 2 .

Now, the group H acts freely on $\prod_{i=1}^s \text{Inc}(V_i, V_{i+1})$, and the quotient is $\mathcal{F}\ell(r_1, \dots, r_s; r)$. Thus we have a natural GL_r -equivariant homomorphism of filtered algebras

$$\eta : \mathfrak{A} \left(\mathcal{D} \left(\bigoplus_{i=1}^s \text{Hom}(V_i, V_{i+1}) \right), \bigoplus_{i=1}^s \mathfrak{gl}_{r_i}, d\mathfrak{J} - \chi \text{Tr} \right) \rightarrow \mathcal{D}_\chi(\mathcal{F}\ell(r_1, \dots, r_s; r)).$$

We are going to show that η is an isomorphism.

First let us show that η is an epimorphism. For this purpose note that by Proposition 2.2.1, we have a surjective map $i : U(\mathfrak{sl}_r) \rightarrow \mathcal{D}_\chi(\mathcal{F}\ell(r_1, \dots, r_s; r))$, and a map

$$\theta : U(\mathfrak{sl}_r) \rightarrow \mathfrak{A} \left(\mathcal{D} \left(\bigoplus_{i=1}^s \text{Hom}(V_i, V_{i+1}) \right), \bigoplus_{i=1}^s \mathfrak{gl}_{r_i}, d\mathfrak{J} - \chi \text{Tr} \right),$$

such that $\eta \circ \theta = i$. This implies that η is surjective.

Now let us show that η is injective. For this purpose, it suffices to show that the associated graded map $\text{gr}(\eta)$ of η is injective. For this, it suffices to prove that the multiplicity $m_1(W)$ of every irreducible finite dimensional GL_r -module W in the source of $\text{gr}\eta$ is at most the multiplicity $m_2(W)$ of this module in the target (note that by Proposition 2.2.1(ii), $m_2(W) = \dim W^L$, where L is the Levi subgroup $\prod_i GL_{r_i - r_{i-1}}$ in GL_r). In our case, the source is the algebra of regular functions on the classical Hamiltonian reduction of the space of representations of the doubled quiver of type A_s with dimension vector (r_1, \dots, r_{s+1}) with respect to the group H (i.e., the last factor GL_r is not included) with the zero value of the moment map. Since in this case the moment map is flat (as follows from Theorem 1.1 in the paper [CB]; it is important here that the last factor is not included), $m_1(W)$ equals the multiplicity of W in the algebra of functions on the classical Hamiltonian reduction as above for a *generic* value of the moment map. But by a Lemma of Crawley-Boevey, this reduction is a semisimple coadjoint orbit of GL_r , isomorphic to GL_r/L (see e.g. [EGO], Lemma 2.6.8). This implies that $m_1(W) = m_2(W)$, and η is an isomorphism, as desired. \square

2.5. Sequence of Reductions. Now we are ready to complete the proof, doing the reduction prescribed in Theorem 2.3.3 (for orientation of all edges towards the node) step by step according to Proposition 1.1.2. Note that separating the node and the “legs” of our graph we have $\mathfrak{gl}(\alpha^{CM}) \cong \mathfrak{gl}_{n_\ell} \oplus$

$\mathfrak{gl}(\alpha^{CM})'$, where

$$\mathfrak{gl}(\alpha^{CM})' \cong \bigoplus_{i=1}^m \mathfrak{g}_i, \quad \mathfrak{g}_i = \bigoplus_{j=1}^{d_i-1} \mathfrak{gl}_{\alpha_{(i,j)}^{CM}}.$$

Proposition 2.5.1. *We have in the notation of the Main Theorem a $\mathfrak{gl}_{n\ell}$ -equivariant isomorphism*

$$\mathfrak{A}(\mathcal{D}(\text{Rep}_{\alpha^{CM}}(Q_{CM})), \mathfrak{gl}(\alpha^{CM})', d\mathfrak{J} - \chi\text{Tr}) \cong U_{\mu_1}^{\mathfrak{p}_1}(\mathfrak{sl}_{n\ell}) \otimes \cdots \otimes U_{\mu_m}^{\mathfrak{p}_m}(\mathfrak{sl}_{n\ell}).$$

Proof. We have

$$\text{Rep}_{\alpha^{CM}}(Q_{CM}) \cong \bigoplus_{i=1}^m \text{Rep}_{\alpha^{CM}}(Q_{CM})_i,$$

such that \mathfrak{g}_i acts only on $\text{Rep}_{\alpha^{CM}}(Q_{CM})_i$. Therefore the left hand side is isomorphic to

$$\bigotimes_{i=1}^m \mathfrak{A}(\mathcal{D}(\text{Rep}_{\alpha^{CM}}(Q_{CM}))_i, \mathfrak{gl}_i, d\mathfrak{J} - \chi\text{Tr}).$$

Combining for each factor Theorem 2.4.1 with Theorem 2.2.1, we obtain the statement of the Proposition. \square

At last note that $\mathfrak{gl}_{n\ell} \cong \mathbb{C} \oplus \mathfrak{sl}_{n\ell}$ and that \mathbb{C} acts on the right hand side by a scalar, so the result of reduction with respect to \mathbb{C} is either zero (not in our case) or the initial algebra. Applying the reduction with respect to $\mathfrak{sl}_{n\ell}$, we obtain the statement of the Main Theorem.

3. CONSTRUCTION OF REPRESENTATIONS

Corollary 1.4.2 together with Proposition 1.1.2 immediately provides a way of constructing many finite dimensional representations of spherical symplectic reflection algebras of types D_4, E_6, E_7, E_8 , by taking U_i to be finite dimensional representations. Such representations are mostly new, and would be interesting to study in detail, which we plan to do in the future. But they defined only for a certain discrete set of values of parameters. Below we construct the finite dimensional representations which are defined on hyperplanes (of codimension 1) in the space of parameters. These representations were discovered in [EM].

3.1. Some isomorphisms of enveloping algebra quotients. Let us establish some isomorphisms between different $U_{\mu}^{\mathfrak{p}}(\mathfrak{g})$, to be used in the next subsection, (they are a peculiarity of $\mathfrak{g} = \mathfrak{sl}_r$). Let \mathfrak{p}_1 be a parabolic subalgebra of \mathfrak{sl}_r and μ_1 a character of \mathfrak{p}_1 . Let m_1, \dots, m_k be the sizes of blocks of the Levi subalgebra $\mathfrak{l}_1 \subset \mathfrak{p}_1$. Let \mathfrak{p}_2 be the parabolic subalgebra with the sizes of blocks $m_1, \dots, m_{i+1}, m_i, \dots, m_k$ (two neighboring blocks transposed). Consider the permutation $\sigma \in S_r$ which is the transposition of the i -th and the $i+1$ -th blocks.

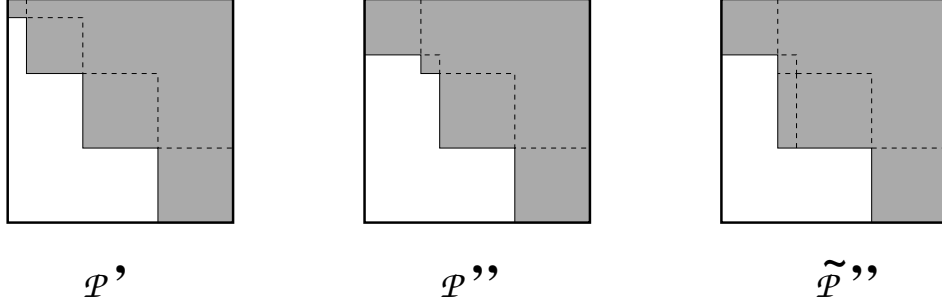
The following proposition is known, but for reader's convenience we include a proof (based on quantum reduction) in the appendix to this paper.

Proposition 3.1.1. *We have $U_{\mu_1}^{\mathfrak{p}_1}(\mathfrak{g}) \cong U_{\mu_2}^{\mathfrak{p}_2}(\mathfrak{g})$ with $\mu_2 = \sigma(\mu_1 + \rho) - \rho$, where ρ is the half-sum of positive roots.*

More generally, let \mathfrak{p}_1 and \mathfrak{p}_2 be parabolic subalgebras of \mathfrak{sl}_r with Levi subalgebras \mathfrak{l}_1 and \mathfrak{l}_2 . By ρ_1 and ρ_2 denote the half-sums of positive roots of \mathfrak{l}_1 and \mathfrak{l}_2 respectively. Suppose that $\mathfrak{l}_1 \cong \mathfrak{l}_2$, then there exists a “block-wise” permutation σ such that $\sigma(\rho_1) = \rho_2$. An advantage of this permutation is that for a weight $\mu_1 : \mathfrak{p}_1 \rightarrow \mathbb{C}$ we have that $\sigma(\mu_1 + \rho) - \rho$ is a weight of \mathfrak{p}_2 .

Corollary 3.1.2. *We have $U_{\mu_1}^{\mathfrak{p}_1}(\mathfrak{g}) \cong U_{\mu_2}^{\mathfrak{p}_2}(\mathfrak{g})$ with $\mu_2 = \sigma(\mu_1 + \rho) - \rho$.*

Now let $\mathfrak{p}''(s, r)$ be the parabolic subalgebra in $\mathfrak{g} = \mathfrak{sl}_r$ generated by upper triangular matrices and the elements and $f_i = E_{i+1, i}$ for $i \neq \frac{r}{s} - 1$ and $\frac{r}{s}$ not dividing i . Also, denote by $\tilde{\mathfrak{p}}''(s, r)$ the subalgebra generated by $\mathfrak{p}''(s, r)$ and $f_{r/s} = E_{\frac{r}{s}+1, \frac{r}{s}}$.



Corollary 3.1.3. *We have $U_{\nu}^{\mathfrak{p}''(s, r)}(\mathfrak{g}) \cong U_{\mu}^{\mathfrak{p}'(s, r)}(\mathfrak{g})$, where*

$$\nu^1 = 0, \quad \mu^{r/s-1} = 0, \quad \nu^{r/s-1} = -\mu^1 - r/s, \quad \nu^{r/s} = \mu^1 + \mu^{r/s} + r/s - 1$$

and $\nu^i = \mu^i$ for all other i .

Proof. The corollary follows from Proposition 3.1.1. □

3.2. The open orbit lemma. Let $G = PGL_{n\ell}(\mathbb{C})$. Let P_i be the parabolic subgroups of G corresponding to the Lie algebras \mathfrak{p}_i , introduced in Theorem 1.4.1. Let $\tilde{P}''(s, r)$ be the parabolic subgroup of G corresponding to the Lie algebra $\tilde{\mathfrak{p}}''(s, r)$.

Lemma 3.2.1. *The group G has an open orbit X_* on the space*

$$X := G/P_1 \times \dots \times G/P_{m-1} \times G/\tilde{P}''(\ell, n\ell),$$

and the action of G on this orbit is free.

Proof. The result follows from the fact that for any real positive root β of a simply laced affine root system, a generic representation of the corresponding affine quiver with dimension vector β has one-dimensional endomorphism algebra (in particular, is indecomposable), and such a representation is unique

(Kac's theorem, [Ka]). Namely, in our case, $\beta = n\delta - \alpha_0 = (n-1)\delta + \theta$, where θ is the maximal root of the corresponding finite root system. So β is a real root and we can apply Kac's theorem. \square

3.3. A representation of $U_{\mu_1}^{\mathfrak{p}_1} \otimes \dots \otimes U_{\mu_m}^{\mathfrak{p}_m}$. Let $\tilde{\mathfrak{p}} \supset \mathfrak{p}$ be the parabolic subalgebra of \mathfrak{g} generated by \mathfrak{p} and f_α for some simple root α . Suppose that ν is a character of \mathfrak{p} , and $\langle \nu, \alpha \rangle \in \mathbb{Z}_{\geq 0}$. Then the irreducible highest weight representation L_ν of the Levi subgroup of $\tilde{L} \subset \tilde{P}$ is finite dimensional. The epimorphism $\tilde{P} \rightarrow \tilde{L}$ defines the structure of a \tilde{P} -module on L_ν .

Now let ν_m be the weight related to μ_m as in Corollary 3.1.3. Assume that $\nu_m^n := \langle \nu_m, \alpha_n \rangle$ is a positive integer, and let $L := L_{\nu_m}$ be the corresponding finite dimensional module over $\tilde{\mathfrak{p}}_{\ell, n\ell}$. Then L is naturally a representation of $\hat{\mathfrak{p}} := \mathfrak{p}_1 \times \dots \times \mathfrak{p}_{m-1} \times \tilde{\mathfrak{p}}''(\ell, n\ell)$, with \mathfrak{p}_i acting through the characters μ_i .

Let $x \in X_*$ be a point in the open orbit, and x' a preimage of X in G^m , and B the formal neighborhood of x' in G^m . Let Y' be the space of regular functions on B with values in L , and Y be the space of $f \in Y'$ such that for any $z \in \hat{\mathfrak{p}}$, $R_z f = -\pi_L(z)f$, where R_z is the vector field of right translation by z . Thus, by Lemma 3.2.1, Y can be identified with the space of L -valued regular functions on the neighborhood of 1 in G , i.e. $(S\mathfrak{g})^* \otimes L$.

Proposition 3.3.1. *The action of \mathfrak{g}^m on Y by left translations makes it into a representation of $U_{\mu_1}^{\mathfrak{p}_1} \otimes \dots \otimes U_{\mu_m}^{\mathfrak{p}_m}$.*

Proof. By construction, Y is a representation of $U_{\mu_1}^{\mathfrak{p}_1} \otimes \dots \otimes U_{\mu_{m-1}}^{\mathfrak{p}_{m-1}} \otimes U_{\nu_m}^{\mathfrak{p}''(\ell, n\ell)}$. So the statement follows from Corollary 3.1.3. \square

3.4. Application to $eH_{1,k,c}e$. Let \mathfrak{p}_i be as in Theorem 1.4.1. Fix weights $\mu_i : \mathfrak{p}_i \rightarrow \mathbb{C}$, with $\nu_m^n := \langle \nu_m, \alpha_n \rangle$ being a nonnegative integer q (where ν_i are related to μ_i as in Corollary 3.1.3). Let k, c be related to μ_i as in Theorem 1.4.1. Let Y be the representation constructed in the previous subsection.

Theorem 3.4.1. *The space $Y^{\mathfrak{g}}$ (the invariants in Y under the diagonal \mathfrak{g} -action) is a $eH_{1,k,c}e$ -module of dimension $\binom{n+q}{n}$.*

Proof. The fact that $Y^{\mathfrak{g}}$ is a $eH_{1,k,c}e$ -module follows from Corollary 1.4.2. The dimension formula follows from Lemma 3.2.1 (which implies that \mathfrak{g} acts simply transitively on the formal neighborhood of x). Indeed, the lemma implies that $Y^{\mathfrak{g}}$ can be identified with L , and the dimension of L is $\binom{n+q}{n}$, since $L = S^q \mathbb{C}^{n+1}$. \square

Remark 3.4.2. Let x_i be the projection of x to the i -th factor of the product $G/P_1 \times \dots \times G/P_{m-1} \times G/\tilde{P}''(\ell, n\ell)$, and H_i be the space of L^* -valued distributions on G concentrated on the preimage of x_i , satisfying the condition $R_z \psi = -\pi_{L^*}(z)\psi$, $z \in \mathfrak{p}_i$. Then $Y = (H_1 \otimes \dots \otimes H_m)^*$, hence $Y^{\mathfrak{g}} = ((H_1 \otimes \dots \otimes H_m)_{\mathfrak{g}})^*$. Note that H_i are generalized Verma modules for appropriate polarizations of \mathfrak{g} .

Note that for generic parameters k and c there are no finite-dimensional representations of $eH_{1,k,c}e$, they appear on hyperplanes (see [EM]). Theorem 3.4.1 constructs them for a generic set of parameters at hyperplanes

$$\nu_m^n = \lambda(c)_o + \frac{k(n-1)}{2} - 1 \in \mathbb{Z}_+.$$

4. APPENDIX: PROOF OF PROPOSITION 3.1.1

First, we show this in the case of 2 blocks of sizes m_1, m_2 . In this case, the space of characters $\mathfrak{p}_1 \rightarrow \mathbb{C}$ is one-dimensional, hence we can assume that μ_1 is a complex number such that $e_{ij} \mapsto \frac{m_2\mu_1}{m_1+m_2}\delta_{ij}$ for $i = 1, \dots, m_1$ and $e_{ij} \mapsto \frac{-m_1\mu_1}{m_1+m_2}\delta_{ij}$ for $i = m_1 + 1, \dots, m_1 + m_2$. We have

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & m_1 & m_1 + 1 & \dots & m_1 + m_2 \\ m_2 + 1 & m_2 + 2 & \dots & m_2 + m_1 & 1 & \dots & m_2 \end{pmatrix},$$

and $\mu_2 = \sigma(\mu_1 + \rho) - \rho = -\mu_1 - m_1 - m_2$.

Let V be the tautological representation of $\mathfrak{sl}_r = \mathfrak{sl}_{m_1+m_2}$. By Theorem 2.2.1, we have $U_{\mu_1}^{\mathfrak{p}_1}(\mathfrak{g}) \cong \mathcal{D}_{\mu_1}(\mathcal{F}\ell(m_1, m_1 + m_2))$, where

$$\mathcal{F}\ell(m_1, m_1 + m_2) = \text{Inc}(\mathbb{C}^{m_1}, V)/GL_{m_1}$$

is the Grassmann variety. Due to Theorem 2.4.1 we have

$$\mathcal{D}_{\mu_1}(\mathcal{F}\ell(m_1, m_1 + m_2)) = \mathfrak{A}(\mathcal{D}(\mathbb{C}^{m_1} \otimes V), \mathfrak{gl}_{m_1}, d\phi - \mu_1 \text{Tr}),$$

where \mathfrak{gl}_{m_1} acts on the first tensor factor, and the map $d\phi - \mu_1 \text{Tr}$ is defined as follows. Let $v_{i,k}$, $i = 1, \dots, m_1$, $k = 1, \dots, m_1 + m_2$, be a basis of $\mathbb{C}^{m_1} \otimes V$, and $v_{i,k}^*$ be the dual basis of $\mathbb{C}^{m_1} \otimes V^*$, then

$$(d\phi - \mu_1 \text{Tr})(e_{ij}) = \sum_{k=1}^{m_1+m_2} v_{j,k} \partial_{v_{i,k}} - \mu_1 \delta_{i,j} \quad : \quad \mathfrak{gl}_{m_1} \rightarrow \mathcal{D}(\mathbb{C}^{m_1} \otimes V).$$

Let $\mathfrak{F} : \mathcal{D}(\mathbb{C}^{m_1} \otimes V) \xrightarrow{\sim} \mathcal{D}(\mathbb{C}^{m_1} \otimes V^*)$ be the *Fourier transform*, which sends $v_{i,k}$ to $\partial_{v_{i,k}^*}$, and $\partial_{v_{i,k}}$ to $-v_{i,k}^*$. We have

$$\begin{aligned} \mathfrak{F}((d\phi - \mu_1 \text{Tr})(e_{ij})) &= \mathfrak{F}\left(\sum_{k=1}^{m_1+m_2} v_{j,k} \partial_{v_{i,k}} - \mu_1 \delta_{i,j}\right) = \\ &= \sum_{k=1}^{m_1+m_2} -\partial_{v_{i,k}^*} v_{j,k}^* - \mu_1 \delta_{i,j} = \sum_{k=1}^{m_1+m_2} -v_{j,k}^* \partial_{v_{i,k}^*} - (\mu_1 + m_1 + m_2) \delta_{i,j} = \\ &= -(d\phi - \mu_2 \text{Tr})(e_{ji}). \end{aligned}$$

Thus, the Fourier transform gives an isomorphism

$$\begin{aligned} \mathfrak{A}(\mathcal{D}(\mathbb{C}^{m_1} \otimes V), \mathfrak{gl}_{m_1}, d\phi - \mu_1 \text{Tr}) &\cong \mathfrak{A}(\mathcal{D}(\mathbb{C}^{m_1} \otimes V^*), \mathfrak{gl}_{m_1}, d\phi - \mu_2 \text{Tr}) = \\ &= \mathcal{D}_{\mu_2}(\text{Inc}(\mathbb{C}^{m_1}, V^*)/GL_{m_1}). \end{aligned}$$

Note that $\text{Inc}(\mathbb{C}^{m_1}, V^*)/GL_{m_1}$ is naturally identified with the space of surjective operators $V^* \rightarrow \mathbb{C}^{m_2}$ up to the GL_{m_2} -action on the target space,

and the latter is naturally $\text{Inc}(\mathbb{C}^{m_2}, V)/GL_{m_2}$ (this is an isomorphism of algebraic varieties). Thus we have

$$\begin{aligned} U_{\mu_1}^{\mathfrak{p}_1}(\mathfrak{g}) &= \mathfrak{A}(\mathcal{D}(\mathbb{C}^{m_1} \otimes V), \mathfrak{gl}_{m_1}, d\phi - \mu_1 \text{Tr}) \cong \\ &\cong \mathcal{D}_{\mu_2}(\text{Inc}(\mathbb{C}^{m_2}, V)/GL_{m_2}) = \mathcal{D}_{\mu_2}(\mathcal{F}\ell(m_2, m_1 + m_2)) = U_{\mu_2}^{\mathfrak{p}_2}(\mathfrak{g}). \end{aligned}$$

Note that all the isomorphisms above preserve the action of $\mathfrak{gl}_r = \mathfrak{gl}_{m_1+m_2}$. This enables us to pass to the general case as follows. Consider the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ generated by the subalgebra of upper-triangular matrices and the Levi subalgebra of block-diagonal matrices with the sizes of blocks equal to $m_1, \dots, m_{i-1}, m_i + m_{i+1}, \dots, m_k$. Let $P \subset G$ be the corresponding parabolic subgroup. The variety

$$SL_r/P_1 = \mathcal{F}\ell(m_1, m_1 + m_2, \dots, m_1 + \dots + m_{k-1}; r)$$

can be regarded as the homogeneous bundle $SL_r *_P \mathcal{F}\ell(m_i; m_i + m_{i+1})$, where P acts on $\mathcal{F}\ell(m_i; m_i + m_{i+1})$ through the homomorphism $P \rightarrow GL_{m_i+m_{i+1}}$. Respectively, we have

$$SL_r/P_2 = SL_r *_P \mathcal{F}\ell(m_{i+1}; m_i + m_{i+1}).$$

Consider the homogeneous bundle $SL_r *_{[P,P]} (\mathbb{C}^{m_i} \otimes V)$, where V is the tautological representation of $SL_{m_i+m_{i+1}}$, and P acts on V through the homomorphism $[P, P] \rightarrow SL_{m_i+m_{i+1}}$. Suppose

$$\mu_1 = \sum_{s=1}^{k-1} \mu_1^{(s)} \text{Tr}_s \quad \text{where} \quad \text{Tr}_s(p) = \text{Tr} \left(p \sum_{i=m_1+\dots+m_{s-1}+1}^{m_1+\dots+m_s} e_{ii} \right).$$

Then the algebra $U_{\mu_1}^{\mathfrak{p}_1}(\mathfrak{g}) = \mathcal{D}_{\mu_1}(SL_r *_P \mathcal{F}\ell(m_i; m_i + m_{i+1}))$ is the following quantum Hamiltonian reduction:

$$\mathfrak{A}(\mathcal{D}(SL_r *_{[P,P]} (\mathbb{C}^{m_i} \otimes V)), \mathfrak{gl}_{m_i} \oplus \mathfrak{p}/[\mathfrak{p}, \mathfrak{p}], (d\phi - (\mu_1^{(i)} - \mu_1^{(i+1)}) \text{Tr}) \oplus (\mathcal{E} - \bar{\mu}_1^{(i)})),$$

where $\bar{\mu}_1^{(i)} = \sum_{s \neq i, i+1} \mu_1^{(s)} \text{Tr}_s + \frac{\mu_1^{(i)} m_i + \mu_1^{(i+1)} m_{i+1}}{m_i + m_{i+1}} (\text{Tr}_i + \text{Tr}_{i+1})$. Here GL_{m_i} acts on \mathbb{C}^{m_i} , and $P/[P, P]$ acts on the homogeneous bundle by right shifts. Note that $\bar{\mu}_1^{(i)} = \bar{\mu}_2^{(i)}$ and $\mu_2^{(i)} - \mu_2^{(i+1)} = -(\mu_1^{(i)} - \mu_1^{(i+1)}) - m_i - m_{i+1}$.

Since the Fourier transform $\mathfrak{F} : \mathcal{D}(\mathbb{C}^{m_i} \otimes V) \xrightarrow{\sim} \mathcal{D}(\mathbb{C}^{m_i} \otimes V^*)$ is $\mathfrak{gl}_{m_1+m_2}$ -invariant, there is a well-defined *fiberwise Fourier transform*

$$\mathfrak{F}_P : \mathcal{D}(SL_r *_{[P,P]} (\mathbb{C}^{m_i} \otimes V)) \xrightarrow{\sim} \mathcal{D}(SL_r *_{[P,P]} (\mathbb{C}^{m_i} \otimes V^*)).$$

The computation in the case of two blocks shows that this isomorphism gives an isomorphism of quantum Hamiltonian reductions

$$\begin{aligned} &\mathfrak{A}(\mathcal{D}(SL_r *_{[P,P]} (\mathbb{C}^{m_i} \otimes V)), \mathfrak{gl}_{m_i} \oplus \mathfrak{p}/[\mathfrak{p}, \mathfrak{p}], (d\phi - \mu_1^{(i)} \text{Tr}) \oplus (\mathcal{E} - \bar{\mu}_1^{(i)})) \cong \\ &\cong \mathfrak{A}(\mathcal{D}(SL_r *_{[P,P]} (\mathbb{C}^{m_i} \otimes V^*)), \mathfrak{gl}_{m_i} \oplus \mathfrak{p}/[\mathfrak{p}, \mathfrak{p}], (d\phi - \mu_2^{(i)} \text{Tr}) \oplus (\mathcal{E} - \bar{\mu}_1^{(i)})). \end{aligned}$$

The right-hand side is naturally isomorphic to

$$\begin{aligned} \mathfrak{A}(\mathcal{D}(SL_r *_{[P,P]}(\mathbb{C}^{m_{i+1}} \otimes V)), \mathfrak{gl}_{m_{i+1}} \oplus \mathfrak{p}/[\mathfrak{p}, \mathfrak{p}], (d\phi - \mu_2^{(i)} \text{Tr}) \oplus (\mathcal{E} - \bar{\mu}_2^{(i)})) = \\ = \mathcal{D}_{\mu_2}(SL_r *^P \mathcal{F}\ell(m_{i+1}; m_i + m_{i+1})) = U_{\mu_2}^{\mathfrak{p}_2}(\mathfrak{g}). \end{aligned}$$

Proposition 3.1.1 is proved.

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